511.64
M22s

Mac Mahon, P.A.

The superior and inferior indices of permutations

UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
MATHEMATICS

The person charging this material is responsible for its return to the library from which it was withdrawn on or before the **Latest Date** stamped below.

Theft, mutilation, and underlining of books are reasons for disciplinary action and may result in dismissal from the University.

UNIVERSITY OF ILLINOIS LIBRARY AT URBANA-CHAMPAIGN

OCT 1 1974
OCT 19 1974
OCT 22 RECD

L161—O-1096

512.5
M22s
514.64

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY

VOLUME XXII. No. IV. pp. 55—60.

THE SUPERIOR AND INFERIOR INDICES OF PERMUTATIONS

BY

MAJOR P. A. MACMAHON, F.R.S.

HONORARY MEMBER CAMBRIDGE PHILOSOPHICAL SOCIETY

CAMBRIDGE:

AT THE UNIVERSITY PRESS

AND SOLD BY

DEIGHTON, BELL AND CO. AND BOWES AND BOWES, CAMBRIDGE.
CAMBRIDGE UNIVERSITY PRESS, LONDON.

M.DCCCC.XIV.

Price One Shilling and Sixpence.

4 *April*, 1914.

LIBRARY
UNIVERSITY OF ILLINOIS
URBANA

Digitized by the Internet Archive
in 2021 with funding from
University of Illinois Urbana-Champaign

https://archive.org/details/superiorinferior00macm

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY

VOLUME XXII. No. IV. pp. 55—60.

THE SUPERIOR AND INFERIOR INDICES OF PERMUTATIONS

BY

MAJOR P. A. MACMAHON, F.R.S.

HONORARY MEMBER CAMBRIDGE PHILOSOPHICAL SOCIETY

CAMBRIDGE:
AT THE UNIVERSITY PRESS

M.DCCCC.XIV.

ADVERTISEMENT

The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Volume of the Transactions.

PRINTED IN GREAT BRITAIN.

512.5
M22
MATHEMATICS LIBRARY

IV. *The Superior and Inferior Indices of Permutations.*

By Major P. A. MacMahon, F.R.S., Hon. Member Camb. Phil. Soc.

[*Received* 12 January 1914.]

Reference is made to the paper on Indices of Permutations*.

I here define indices of a new kind and subject them to investigation.

Let any assemblage of letters be

$$\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_s^{i_s}$$

and consider any permutation of them.

If any letter precedes p_1' letters which have a smaller subscript we obtain the component p_1' of the Superior Index.

The Superior Index of the permutation is defined to be

$$\Sigma p_1' = p',$$

the summation being in respect to every letter of the permutation.

On the other hand if any letter precedes r_1' letters which have a larger subscript we obtain the component r_1' of the Inferior Index.

The Inferior Index is defined to be

$$\Sigma r_1' = r',$$

the summation being in respect to every letter of the permutation.

Ex. gr. Consider the assemblage $\alpha^4\beta^3\gamma^2\delta$ and the permutation

$\beta\alpha\alpha\delta\gamma\alpha\alpha\beta\beta\gamma$

4 64 , $4+6+4=14$ the Superior Index

355 3311 , $3+5+5+3+3+1+1=21$ the Inferior Index.

If the permutation be reversed

$\gamma\beta\beta\alpha\alpha\gamma\delta\alpha\alpha\beta$

744 33 , $7+4+4+3+3=21$ the Superior Index

122331 11 , $1+2+2+3+3+1+1+1=14$ the Inferior Index,

and we see that the Superior and Inferior Indices of the permutation are respectively equal to

* "The Indices of Permutations and the Derivation therefrom of Functions of a Single Variable associated with the Permutations of any Assemblage of Objects." *American Journal of Mathematics*, Vol. xxxv. No. 3, 1913.

48068

the Inferior and Superior Indices of the reversed permutation. This is obviously true in general so that we can assert in regard to the assemblage

$$\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_s^{i_s}$$

that the collection of numbers which specifies the superior indices of the permutations is identical with the collection which specifies the inferior indices.

Hence, in regard to the permutations of any assemblage,

$$\Sigma x^{p'} = \Sigma x^{r'}.$$

It is also readily established that, for every permutation,

$$p' + r' = \Sigma i_1 i_2;$$

for consider that part of a permutation which involves two letters α_k, α_m.

Suppose it to be

$$\dots \alpha_k^{i_k'} \dots \alpha_m^{i_m'} \dots \alpha_m^{i_m''} \dots \alpha_k^{i_k''} \dots \alpha_k^{i_k'''} \dots \alpha_m^{i_m'''} \dots.$$

The portion of the superior index due to these two letters is, if $k > m$,

$$i_k'(i_m' + i_m'' + i_m''' + \dots) + (i_k'' + i_k''')(i_m''' + \dots)$$

and the portion of the inferior index is

$$(i_m' + i_m'')(i_k'' + i_k''' + \dots) + i_m'''(\dots).$$

Adding these together we find that the two letters contribute to the sums of the two indices the number

$$i_m i_k.$$

Thence obviously $$p' + r' = \Sigma i_1 i_2,$$

leading to the relation $$\Sigma x^{p'} = x^{\Sigma i_1 i_2}\Sigma x^{-p'}.$$

The maximum value of p' is clearly $\Sigma i_1 i_2$ and its average value $\frac{1}{2}\Sigma i_1 i_2$.

The function $\Sigma x^{p'}$ is of degree $\Sigma i_1 i_2$ in x and if it be divided by $x^{\frac{1}{2}\Sigma i_1 i_2}$ it is unaltered by the substitution of $\frac{1}{x}$ for x because

$$\Sigma x^{p' - \frac{1}{2}\Sigma i_1 i_2} = \Sigma x^{\frac{1}{2}\Sigma i_1 i_2 - p'}.$$

A function of x which satisfies these conditions is

$$\frac{(1-x)(1-x^2)\dots\dots(1-x^{i_1+i_2+\dots+i_s})}{(1-x)(1-x^2)\dots(1-x^{i_1})\cdot(1-x)(1-x^2)\dots(1-x^{i_2})\dots\dots(1-x)(1-x^2)\dots(1-x^{i_s})}$$

and it will be shewn that this is in fact equal to $\Sigma x^{p'}$.

In the first place consider the assemblage $\alpha^i\beta^j$, and write $\Sigma x^{p'} = F_x(i, j)$.

All permutations which terminate with β, contribute

$$F_x(i, j-1)$$

to $F_x(i, j)$, and those which terminate with α, contribute

$$x^j F(i-1, j).$$

Hence the difference equation

$$F_x(i, j) = x^j F_x(i-1, j) + F_x(i, j-1),$$

the solution of which, satisfying the above conditions, is

$$F_x(i, j) = \frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i+j})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}).(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})}$$

where $(\mathbf{m})$ has been written to denote (after Cayley) $1 - x^m$.

Similarly for the assemblage $\alpha^i\beta^j\gamma^k$, write $\Sigma x^{p'} = F_x(i, j, k)$. The permutations terminating with γ, β, α respectively contribute

$$F_x(i, j, k-1),\ x^k F_x(i, j-1, k) \text{ and } x^{j+k} F_x(i-1, j, k-1) \text{ to } F_x(i, j, k),$$

leading us to the difference equation

$$F_x(i, j, k) = x^{j+k} F_x(i-1, j, k) + x^k F_x(i, j-1, k) + F_x(i, j, k-1),$$

the solution of which, satisfying the conditions, is

$$F_x(i, j, k) = \frac{(\mathbf{1})\dots(\mathbf{i+j+k})}{(\mathbf{1})\dots(\mathbf{i}).(\mathbf{1})\dots(\mathbf{j}).(\mathbf{1})\dots(\mathbf{k})}.$$

Similarly we reach the difference equation

$$F_x(i_1, i_2, i_3 \dots i_s) = x^{i_2+i_3+\dots+i_s} F_x(i_1 - 1, i_2, \dots i_s) + x^{i_3+\dots+i_s} F_x(i_1, i_2 - 1, \dots i_s) + \dots + F_x(i_1, i_2, \dots i_s - 1),$$

the solution of which, satisfying the conditions, is

$$F_x(i_1, i_2, \dots i_s) = \frac{(\mathbf{1})\dots(\mathbf{i_1+i_2+\dots+i_s})}{(\mathbf{1})\dots(\mathbf{i_1}).(\mathbf{1})\dots(\mathbf{i_2})\dots\dots(\mathbf{1})\dots(\mathbf{i_s})}.$$

This result is remarkable because it establishes that

$$\Sigma x^{p'} = \Sigma x^p,$$

where p is the Greater Index of a permutation (vide *American Journ. Math.* Vol. XXXV. No. 3, 1913). In fact the whole collection of Superior Indices coincides with the whole collection of Greater Indices, but it is not easy to establish this by a one-to-one correspondence.

Observe the permutations of $\alpha\alpha\beta\gamma$.

Permutation	Greater Index	Superior Index
$\alpha\alpha\beta\gamma$	0	0
$\alpha\alpha\gamma\beta$	3	1
$\alpha\beta\alpha\gamma$	2	1
$\alpha\gamma\alpha\beta$	2	2
$\alpha\beta\gamma\alpha$	3	2
$\alpha\gamma\beta\alpha$	5	3
$\beta\alpha\alpha\gamma$	1	2
$\gamma\alpha\alpha\beta$	1	3
$\beta\alpha\gamma\alpha$	4	3
$\gamma\alpha\beta\alpha$	4	4
$\beta\gamma\alpha\alpha$	2	4
$\gamma\beta\alpha\alpha$	3	5

$$\Sigma x^p = \Sigma x^{p'} = \frac{(\mathbf{1})(\mathbf{2})(\mathbf{3})(\mathbf{4})}{(\mathbf{1})(\mathbf{2}).(\mathbf{1}).(\mathbf{1})} = \frac{(\mathbf{3})(\mathbf{4})}{(\mathbf{1})^2}.$$

The particular result

$$\Sigma x^{p'} = \frac{(1)(2)\dots(i+j)}{(1)\dots(i).(1)\dots(j)}$$

establishes that the permutations of the assemblage $\alpha^i\beta^j$ which have a superior (or greater) index equal to p' are equinumerous with the partitions of the number p' into parts, not exceeding i in magnitude and not exceeding j in number.

The property of $\Sigma x^{p'}$ that is before us leads to interesting relations between the functions $F_x(i, j)$.

Write the assemblage $\alpha^i\beta^j$ in the form

$$\alpha^{i-a}\beta^{j-b}.\alpha^a\beta^b$$

wherein a, b are any two numbers, such that $a \not> i$, $b \not> j$.

It is to be shewn that

$$F_x(i, j) = \Sigma x^{(j-b)a} F_x(i-a, j-b) F_x(a, b)$$

wherein the summation is in respect of every composition a, b of the constant number $a+b$. The number zero is not excluded so that if for instance $a+b=4$, the summation will be in respect of the compositions 40, 31, 22, 13, 04, it being understood, as above stated, that $a \not> i$, $b \not> j$.

It will be admitted that when the permutations admit of representation in the form

Some permutation of $\alpha^{i-a}\beta^{j-b}$ followed by some permutation of $\alpha^a\beta^b$,

the expression $x^{(j-b)a} F_x(i-a, j-b) F_x(a, b)$ denotes $\Sigma x^{p'}$ for the permutations in question. If we sum this expression for all values of a and b which give permutations involving $i+j-a-b$ letters followed by permutations involving $a+b$ letters we must arrive at the expression of $\Sigma x^{p'}$ for the whole of the permutations of $\alpha^i\beta^j$.

Hence $$F_x(i, j) = \Sigma x^{(j-b)a} F_x(i-a, j-b) F_x(a, b)$$

where $a+b =$ any constant number.

This interesting relation between the functions x has a very interesting particular case.

If $a+b=\sigma$ a constant, we have

$$F_x(i, j) = x^{\sigma j} F_x(\sigma, 0) F_x(i-\sigma, j) + x^{(\sigma-1)(j-1)} F_x(\sigma-1, 1) F_x(i-\sigma+1, j-1) + \dots + F_x(0, \sigma) F_x(i, j-\sigma).$$

Putting $\sigma = j$ we obtain

$$F_x(i, j) = x^{j^2} F_x(j, 0) F_x(i-j, j) + x^{(j-1)^2} F_x(j-1, 1) F_x(i-j+1, j-1) + \dots + F_x(0, j) F_x(i, 0)$$

and if we now put $i = j$

$$F_x(j, j) = x^{j^2}\{F_x(j, 0)\}^2 + x^{(j-1)^2}\{F_x(j-1, 1)\}^2 + x^{(j-2)^2}\{F_x(j-2, 2)\}^2 + \dots + \{F_x(0, j)\}^2.$$

This result is a generalization of the theorem in regard to the sum of the squares of the binomial coefficients, for putting $x=1$, it becomes

$$\binom{2j}{j} = \binom{j}{0}^2 + \binom{j}{1}^2 + \binom{j}{2}^2 + \dots + \binom{j}{j}^2.$$

In general the reader will see that we have the relation

$$F_x(i_1 i_2 \dots i_s)$$
$$= \Sigma x^{(i_s - a_s)(a_1 + a_2 + \dots + a_{s-1}) + (i_{s-1} - a_{s-1})(a_1 + a_2 + \dots + a_{s-2}) + \dots + (i_2 - a_2) a_1}$$
$$. F_x(i_1 - a_1,\ i_2 - a_2, \dots i_s - a_s) . F_x(a_1 a_2 \dots a_s),$$

the summation being for every composition of a given number $a_1 + a_2 + \dots + a_s$ into s or fewer parts $a_1, a_2, \dots a_s$ such that $a_s \not> i_s$ for all values of s.

The Superior Index as defined is obtained by adding several numbers together. This is the simplest way of obtaining the index, but the numbers so added are not the most interesting that come up for consideration. If $v > u$ the letter α_v adds a number to the index if it precedes one or more letters α_u. Denote by p'_{vu} the number added to the index due to the positions of the letters α_u, α_v. Moreover α_v may precede 1, 2, ... or i_u letters α_u. Denote by $p'_{vu,\sigma}$ the number of letters α_v which precede exactly σ letters α_u. Every time an α_v precedes exactly σ letters α_u the number σ is added to the index. Hence

$$p'_{vu} = p'_{vu,1} + 2p'_{vu,2} + 3p'_{vu,3} + \dots + i_u p'_{vu,i_u}.$$

Also if p''_{vu} denotes, in regard to the whole of the permutations, the sum of the numbers added to the indices by reason of the relative positions of the letters α_u, α_v and $p''_{vu,\sigma}$ the number of times in the whole of the permutations that a letter α_v precedes exactly σ letters α_u,

$$p''_{vu} = p''_{vu,1} + 2p''_{vu,2} + 3p''_{vu,3} + \dots + i_u p''_{vu,i_u}.$$

Now we know the value of p''_{vu} from the following consideration. In any permutation consider merely the letters α_u, α_v. If r'_{uv} denotes the number added to the Inferior Index by the relative positions of these letters we see that

$$p'_{vu} + r'_{uv} = i_u i_v,$$

for any one letter α_v contributes to the sum of the Superior and Inferior indices the number i_u and therefore the total of i_v letters α_v contributes the number $i_u i_v$. Hence the *average* value of p'_{vu} in a permutation is $\frac{1}{2} i_u i_v$ and thence the number contributed to the Superior Indices of all of the permutations by the relative positions of α_u and α_v is

$$p''_{vu} = \tfrac{1}{2} i_u i_v \frac{(\Sigma i)!}{i_1!\ i_2! \dots i_s!}.$$

It will now be proved that $p''_{vu,\sigma}$ has a value which is independent of the number σ.

Consider the permutations of the assemblage

$$\alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_{u-1}^{i_{u-1}} \alpha_u^{i_u + 1} \alpha_v^{i_v - 1} \alpha_{v+1}^{i_{v+1}} \dots \alpha_s^{i_s},$$

which is derived from the original assemblage by adding an α_u and subtracting an α_v.

In any permutation fix the attention upon the $i_u + 1$ letters α_u. Call the one on the extreme right the last α_u, the one nearest to it the last but one α_u, the next one again the last but two α_u and so on. Now delete the last α_u but σ from the permutation and substitute for it the letter α_v. We have thus an α_v followed by σ letters α_u and the assemblage is the original assemblage of letters. We thus construct a case of an α_v followed by exactly σ letters α_u from every one of the

$$\frac{(\Sigma i)!}{i_1!\, i_2! \dots i_{u-1}!\, (i_u + 1)!\, (i_v - 1)!\, i_{v+1}! \dots i_s!}$$

permutations of the assemblage

$$\frac{\alpha_u}{\alpha_v}\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_s^{i_s}.$$

Thence
$$p''_{vu,\sigma}=\frac{i_v}{i_u+1}\,\frac{(\Sigma i)!}{i_1!\,i_2!\dots i_s!},$$

a value which is independent of σ.

Therefore
$$p''_{vu,0}=p''_{vu,1}=\dots=p''_{vu,i_u},$$

leading to
$$p''_{vu}=\binom{i_u+1}{2}p''_{vu,\sigma}=\tfrac{1}{2}i_u i_v\,\frac{(\Sigma i)!}{i_1!\,i_2!\dots i_s!},$$

and
$$p''_{vu,\sigma}=\frac{i_v}{i_u+1}\,\frac{(\Sigma i)!}{i_1!\,i_2!\dots i_s!}.$$

We deduce that the average value of $p''_{vu,\sigma}$ in a permutation is

$$\frac{i_v}{i_u+1}.$$

To illustrate these results take the assemblage $\alpha\alpha\beta\gamma$ wherein $i_1=2$, $i_2=1$, $i_3=1$.

	$\beta\alpha$	$\gamma\alpha$	$\gamma\beta$
$aa\beta\gamma$	0	0	0
$aa\gamma\beta$	0	0	1
$a\beta a\gamma$	1	0	0
$a\gamma a\beta$	0	1	1
$a\beta\gamma a$	1	1	0
$a\gamma\beta a$	1	1	1
$\beta aa\gamma$	2	0	0
$\gamma aa\beta$	0	2	1
$\beta a\gamma a$	2	1	0
$\gamma a\beta a$	1	2	1
$\beta\gamma aa$	2	2	0
$\gamma\beta aa$	2	2	1

Here
$$p''_{21,0}=p''_{21,1}=p''_{21,2}=\frac{1}{3}\,\frac{4!}{2!\,1!\,1!}=4,$$

and we verify that in the first column the numbers 0, 1 and 2 each occur 4 times.

Also
$$p''_{21}=\binom{3}{2}.4=12,$$

and we verify that the sum of the numbers in the first column is 12.

Again
$$p''_{31,0}=p''_{31,1}=p''_{31,2}=\frac{1}{3}.12=4,$$

$$p''_{31}=\binom{3}{2}.4=12,$$

and we verify that in the second column the numbers 0, 1 and 2 each occur 4 times and that the sum of the numbers is 12.

Again
$$p''_{32,0}=p''_{32,1}=\frac{1}{2}.12=6,$$

$$p''_{32}=\binom{2}{2}6=6,$$

and we verify that in the third column the numbers 0 and 1 each occur 6 times and that the sum of the numbers is 6.

UNIVERSITY OF ILLINOIS-URBANA
511.64M22S C001
THE SUPERIOR AND INFERIOR INDICES OF PER
3 0112 017021236